Groups quasi-isometric to symmetric spaces

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Abstract

We determine the structure of finitely generated groups which are quasi-isometric to symmetric spaces of noncompact type, allowing Euclidean de Rham factors If X is a symmetric space of noncompact type with no Euclidean de Rham factor, and Γ is a finitely generated group quasi-isometric to the product $\mathbb{E}^k \times X$, then there is an exact sequence $1 \to H \to \Gamma \to L \to 1$ where H contains a finite index copy of $\mathbb{Z}^k$ and L is a uniform lattice in the isometry group of X. [1]

1 Introduction

The main result of this paper is the following theorem.

Theorem 1.1 *Let X be a symmetric space of noncompact type with no Euclidean de Rham factor, and let Nil be a simply connected nilpotent Lie group equipped with a left-invariant Riemannian metric. Suppose that Γ is a finitely generated group quasi-isometric to $Nil \times X$. Then there is an exact sequence*

$$1 \longrightarrow H \longrightarrow \Gamma \longrightarrow L \longrightarrow 1 \tag{1}$$

where H is a finitely generated group quasi-isometric to Nil and L is a uniform lattice in the isometry group of X.

In particular, when Nil is the trivial group then Γ is a finite extension of a uniform lattice in $Isom(X)$, and when $Nil \simeq \mathbb{R}^k$ then H is virtually abelian of rank k by [Gro81', Pan83].

Certain cases of theorem 1.1 were proved earlier. [Tuk88, Gro81] determined the structure of groups quasi-isometric to $\mathbb{H}^{n\geq 3}$, [Pan89] handled the other rank 1 cases besides $\mathbb{H}^2$, and [CJ94, Ga92] settled the $\mathbb{H}^2$ case. [Rie93] studied groups quasi-isometric to $\mathbb{H}^2 \times \mathbb{R}$; in this case it was already known [Ger92] that such groups need not admit discrete cocompact isometric actions on $\mathbb{H}^2 \times \mathbb{R}$.

It is an intriguing problem to classify the finitely generated groups quasi-isometric to a given space/group. Some other cases where the classification is known are: free

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groups [Gro87, GDH90], certain nilpotent groups [Gro81', Pan83], non-uniform lattices in rank 1 symmetric spaces [Sch95] and in symmetric spaces without rank 1 factors [Esk96], and fundamental groups of Haken 3-manifolds with nontrivial geometric decomposition [KaLe96].

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Contents

2 Preliminaries

In this section we recall some basic definitions and notation. See [Gro93] for more discussion and background.

Definition 2.1 *A map $f : X \longrightarrow Y$ between metric spaces is an (L, A)* **quasi-isometry** *if for every $x_1, x_2 \in X$*

$$L^{-1}d(x_1, x_2) + A \leq d(x_1, x_2) \leq Ld(x_1, x_2) + A,$$

and for every $y \in Y$ we have $d(y, f(X)) < A$. Two quasi-isometries $f_1, f_2 : X \longrightarrow Y$ are **equivalent** *if $d(f_1, f_2) < \infty$.*

If Γ is a finitely generated group, then any two word metrics on Γ are biLipschitz to one another by $id_\Gamma : \Gamma \to \Gamma$. We will implicitly endow our finitely generated groups with word metrics.

Definition 2.2 *An (L, A)***-quasi-action** *of a group Γ on a metric space Z is a map $\rho : \Gamma \times Z \to Z$ so that $\rho(\gamma, \cdot) : Z \to Z$ is an (L, A) quasi-isometry for every $\gamma \in \Gamma$, $d(\rho(\gamma_1, \rho(\gamma_2, z)), \rho(\gamma_1\gamma_2, z)) < A$ for every $\gamma_1, \gamma_2 \in \Gamma$, $z \in Z$, and $d(\rho(e, z), z) < A$ for every $z \in Z$.*

We will denote the self-map $\rho(\gamma, \cdot) : Z \to Z$ by $\rho(\gamma)$. ρ is **discrete** if for any point $z \in Z$ and any radius $R > 0$, the set of all $\gamma \in \Gamma$ such that $\rho(\gamma, z)$ is contained in the ball $B_R(z)$ is finite. ρ is **cobounded** if Z coincides with a finite tubular neighborhood of the "orbit" $\rho(\Gamma)z \subset Z$ for every z. If ρ is a discrete cobounded quasi-action of a finitely generated group Γ on a geodesic metric space Z, it follows easily that the map $\Gamma \to Z$ given by $\gamma \mapsto \rho(\gamma, z)$ is a quasi-isometry for every $z \in Z$.

Definition 2.3 *Two quasi-actions ρ and ρ' are* **equivalent** *if there exists a constant D so that $d(\rho(\gamma), \rho'(\gamma) < D$ for all $\gamma \in \Gamma$.*

Definition 2.4 *Let ρ and ρ' be a quasi-actions of Γ on Z and Z' respectively, and let $\phi : Z \to Z'$ be a quasi-isometry . Then ρ is* **quasi-isometrically conjugate to ρ' via** *ϕ if there is a D so that $d(\phi \circ \rho(\gamma), \rho'(\gamma) \circ \phi) < D$ for all $\gamma \in \Gamma$.*

Lemma 2.5 *(cf [Gro87, 8.2.K]) Let X be a Hadamard manifold of dimension ≥ 2 with sectional curvature $\leq K < 0$, and let $\partial_\infty X$ denote the geometric boundary of X with the cone topology. Recall that every quasi-isometry $\Phi : X \longrightarrow X$ induces a boundary homeomorphism $\partial_\infty \Phi : \partial_\infty X \to \partial_\infty X$.*

1. *If $\rho : \Gamma \times X \to X$ is a quasi-action on X, then ρ is discrete (respectively cobounded) iff $\partial_\infty \phi$ acts properly discontinuously (respectively cocompactly) on the space of distinct triples in $\partial_\infty X$.*

2. *Given (L, A) there is a D so that if ϕ_k, ψ are (L, A) quasi-isometries, then $\partial_\infty \phi_k$ converges uniformly to $\partial_\infty \psi$ iff $\limsup d(\phi_k x, \psi x) < D$ for every $x \in X$. In particular, if $\phi_1, \phi_2 : X \longrightarrow X$ are (L, A) quasi-isometries with the same boundary mappings, then $d(\phi_1, \phi_2) < D$.*

Proof. Let $\partial^3 X \subset \partial_\infty X \times \partial_\infty X \times \partial_\infty X$ denote the subspace of distinct triples. The uniform negative curvature of X implies that there is a D_0 depending only on K such that
(a) For every $x \in X$ there is a triple $(\xi_1, \xi_2, \xi_3) \in \partial^3 X$ such that $d(x, \overline{\xi_i \xi_j}) < D_0$ for every $1 \leq i \neq j \leq 3$, where $\overline{\xi_i \xi_j}$ denotes the geodesic with ideal endpoints ξ_i, ξ_j. Moreover for every C the set $\{(\xi_1, \xi_2, \xi_3) \mid d(x, \overline{\xi_i \xi_j}) < C \text{ for all } 1 \leq i \neq j \leq 3\}$ has compact closure in $\partial^3 X$.
and
(b) For every $(\xi_1, \xi_2, \xi_3) \in \partial^3 X$ there is a point $x \in X$ so that $d(x, \overline{\xi_i \xi_j}) < D_0$ for each $1 \leq i \neq j \leq 3$. And for every C there is a C' depending only on C and K so that $\{x \in X \mid d(x, \overline{\xi_i \xi_j}) < C \text{ for every } 1 \leq i \neq j \leq 3\}$ has diameter $< C'$.
1 and 2 follow easily from this. $\square$

3 Projecting quasi-actions to the factors

Let Nil and X be as in Theorem 1.1 and decompose X into irreducible factors:

$$X = \prod_{i=1}^{l} X_i \tag{2}$$

Suppose ρ is a quasi-action of the finitely generated group Γ on $Nil \times X$. We denote by $p : Nil \times X \to X$ the canonical projection. By applying [KlLe96, Theorem 1.1.2] to each quasi-isometry $\rho(\gamma)$ we construct quasi-actions ρ_i of Γ on X_i so that

$$d(p \circ \rho(\gamma), \prod_{i=1}^{k} \rho_i(\gamma) \circ p) < D$$

for all $\gamma \in \Gamma$ and some positive constant D.

4 Straightening cocompact quasi-actions on irreducible symmetric spaces

The following result is a direct consequence of [Pan89, Théorème 1] and [KlLe96, Theorem 1.1.3].

Fact 4.1 *Let X be an irreducible symmetric space other than a real or complex hyperbolic space. Then every quasi-action on X is equivalent to an isometric action.*

Proof. Let ρ be a quasi-action of a group Γ on X. By the results just cited, there is an isometry $\bar{\rho}(\gamma)$ at finite distance from the quasi-isometry $\rho(\gamma)$ for every $\gamma \in \Gamma$. This isometry is unique and its distance from $\rho(\gamma)$ is uniformly bounded[2] in terms of the constants of the quasi-action. So $\bar{\rho}$ is an isometric action equivalent to ρ. □

We recall that the real and complex hyperbolic spaces of all dimensions admit quasi-isometries which are not equivalent to isometries [Pan89].

Fact 4.2 *Any cobounded quasi-action ρ on a real or complex hyperbolic space is quasi-isometrically conjugate to an isometric action.*

This result is proven in [Tuk88] in the real-hyperbolic case. Using Pansu's theory of Carnot differentiability one can carry out Tukia's arguments for all rank-one symmetric spaces other than hyperbolic plane, cf. [Pan89, sec. 11]. Another proof for the complex-hyperbolic case can be found in [Chow96].

Fact 4.3 *Let ρ be a cobounded quasi-action of a group Γ on $\mathbb{H}^2$. Then ρ is quasi-isometrically conjugate to a cocompact isometric action of Γ on $\mathbb{H}^2$.*

Proof. We recall that every quasi-isometry $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ induces a quasi-symmetric homeomorphism $\partial_\infty \phi : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^2$, see [TuVa82]; moreover the quasi-symmetry constant of $\partial_\infty \phi$ can be estimated in terms of the quasi-isometry constants of ϕ. Since equivalent quasi-isometries yield the same boundary homeomorphism, every quasi-action ρ on $\mathbb{H}^2$ induces a genuine action $\partial_\infty \rho$ on $\partial_\infty \mathbb{H}^2$ by uniformly quasi-symmetric homeomorphisms.

Let $\bar{\Gamma}$ be the quotient of Γ by the kernel of the action $\partial_\infty \rho$, and let $\pi : \Gamma \to \bar{\Gamma}$ be the canonical epimorphism. If two elements $\gamma_1, \gamma_2 \in \Gamma$ have the same boundary

[2]The uniformity in the rank one case follows lemma 2.5.

map then $d(\rho(\gamma_1), \rho(\gamma_2))$ is uniformly bounded by lemma 2.5. Hence we may obtain a quasi-action $\bar{\rho}$ of $\bar{\Gamma}$ on $\mathbb{H}^2$ by choosing $\gamma \in \pi^{-1}(\bar{\gamma})$ for each $\bar{\gamma} \in \bar{\Gamma}$, and setting $\bar{\rho}(\bar{\gamma}) = \rho(\gamma)$. If $\bar{\tau}$ is an isometric action of $\bar{\Gamma}$ on $\mathbb{H}^2$ and $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ quasi-isometrically conjugates $\bar{\rho}$ into $\bar{\tau}$, then ϕ will quasi-isometrically conjugate ρ into the isometric action $\tau : \Gamma \times \mathbb{H}^2 \to \mathbb{H}^2$ given by $\tau(\gamma) = \bar{\tau}(\pi(\gamma))$. Hence it suffices to treat the case when $\bar{\Gamma} = \Gamma$, and so we will assume that $\partial_\infty \rho$ is an effective action.

Lemma 4.4 *The quasi-action ρ is discrete if and only if the action $\partial_\infty \rho$ on $\partial_\infty \mathbb{H}^2$ is discrete in the compact-open topology.*

Proof. Suppose $\partial_\infty \rho$ is discrete, and let (γ_i) be a sequence in Γ so that $\rho(\gamma_i)$ maps a point $p \in \mathbb{H}^2$ into a fixed ball $B_R(p)$. Then by a selection argument we may assume – after passing to a subsequence if necessary – that there is a quasi-isometry $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ so that for every $q \in \mathbb{H}^2$ we have $\limsup_i d(\rho(\gamma_i)(q), \phi(q)) < D$ for some D. Hence the boundary maps $\partial_\infty \rho(\gamma_i)$ converge to $\partial_\infty \phi$, and so the sequence $\partial_\infty \rho(\gamma_i)$ is eventually constant. Since ρ is effective we conclude that γ_i is eventually constant. Therefore ρ is a discrete quasi-action.

If ρ is a discrete quasi-action on $\mathbb{H}^2$, then $\partial_\infty \rho$ is discrete by lemma 2.5. $\square$

Proof of 4.3 continued.

Case 1: $\partial_\infty \rho$ is discrete. In this case, ρ is a discrete convergence group action (lemma 2.5) and by the work of [CJ94, Ga92], there is a discrete isometric action τ of Γ on $\mathbb{H}^2$ so that $\partial_\infty \rho$ is topologically conjugate to $\partial_\infty \tau$. Since ρ is cobounded, $\partial_\infty \rho$ acts cocompactly on the set of distinct triples of points in $\partial_\infty \mathbb{H}^2$ (lemma 2.5); therefore $\partial_\infty \tau$ also acts cocompactly on the space of triples and so τ is a discrete, cocompact, isometric action of Γ on $\mathbb{H}^2$. We now have two discrete, cobounded, quasi-actions of Γ on $\mathbb{H}^2$, so they are quasi-isometrically conjugate by some quasi-isometry $\psi : \mathbb{H}^2 \to \mathbb{H}^2$.

Case 2: $\partial_\infty \rho$ is nondiscrete. By [Hin90, Theorem 4], $\partial_\infty \rho$ is quasi-symmetrically conjugate to $\partial_\infty \tau$, where τ is an isometric action on $\mathbb{H}^2$. The conjugating quasi-symmetric homeomorphism is the boundary of a quasi-isometry $\psi : \mathbb{H}^2 \to \mathbb{H}^2$, [TuVa82], which quasi-isometrically conjugates $\partial_\infty \rho$ into the isometric action action τ. Applying lemma 2.5 again, we conclude that τ is cocompact. $\square$

Section 3, 4.1, 4.2 and 4.3 imply:

Corollary 4.5 *Let X be a symmetric space of noncompact type without Euclidean factor. Then any cobounded quasi-action on X is quasi-isometrically conjugate to a cocompact isometric action on X.*

5 A Growth estimate for small elements in nondiscrete cocompact subgroups of $Isom(X)$

5.1 Parabolic isometries of symmetric spaces

Let X be a symmetric space of noncompact type, and let $G = Isom(X)$.

An isometry $g \in G$ is **semisimple** if its displacement function δ_g attains its infimum and **parabolic** otherwise.

Lemma 5.1 *Let $A \subset G$ be a finitely generated abelian group all of whose nontrivial elements are parabolic. Then A has a fixed point at infinity.*

Proof. Recall that the nearest point projection to a closed convex subset is well-defined and distance non-increasing. This implies that if C is a non-empty A-invariant closed convex set, then for all displacement functions δ_a, $a \in A$, we have $\inf \delta_a = \inf \delta_a\big|_C$. Hence for all $n \in \mathbb{N}$, the intersection of the sublevel sets $\{p \mid \delta_{a_i}(p) \leq \inf \delta_{a_i} + 1/n\}$ is non-empty and contains a point p_n. We have $\delta_{a_i}(p_n) \to \inf \delta_{a_i}$ for all a_i, and since the isometries a_i are parabolic the sequence $\{p_n\}$ subconverges to an ideal boundary point $\xi \in \partial_{Tits} X$. It follows that the a_i fix ξ. $\square$

Lemma 5.2 *Let $a_1, \ldots, a_k \in Isom(X)$ be commuting parabolic isometries. Then there is a sequence of isometries $\{g_n\} \subset G$ so that for every i the sequence $g_n a_i g_n^{-1}$ subconverges to a semisimple isometry $\bar{a}_i$.*

Proof. ¿From the proof of the previous lemma, there is a sequence of points $\{p_n\} \subset X$ converging to an ideal point ξ so that $\delta_{a_i}(p_n) \to \inf \delta_{a_i}$ for all a_i. Pick isometries $g_n \in G$ such that $g_n \cdot p_n = p_0$. The conjugates $g_n a_i g_n^{-1}$ have the same infimum displacement as a_i. Since

$$\delta_{g_n a_i g_n^{-1}}(p_0) = \delta_{a_i}(p_n) \to \inf \delta_{a_i} \quad ,$$

the $g_n a_i g_n^{-1}$ subconverge to a semisimple isometry. $\square$

We call the isometry $g \neq e$ **purely parabolic** if the identity is the only semisimple element in $\overline{Ad_G(G) \cdot g}$.

5.2 The growth estimate

Proposition 5.3 *Let X be a symmetric space of noncompact type with no Euclidean de Rham factors. Let $\Gamma \subset G = Isom(X)$ be a finitely generated, nondiscrete, cocompact subgroup. Let $U \subset Isom(X)$ be a neighborhood of the identity, and set*

$$f(k) := \#\{g \in \Gamma : |g|_\Gamma < k,\ g \in U\},$$

where $|\cdot|_\Gamma$ denotes a word norm on Γ. Then f grows faster than any polynomial, i.e. for every $d > 0$ $\limsup_{k\to\infty} \frac{f(k)}{k^d} = \infty$.

Proof. Let $\bar{\Gamma}^o$ denote the identity component of the closure of Γ in G.
Case 1: $\bar{\Gamma}^o$ is nilpotent. Let A be the last non-trivial subgroup in the derived series of $\bar{\Gamma}^o$. Then $A \subset \bar{\Gamma}$ is a connected abelian subgroup of positive dimension, A is normal in $\bar{\Gamma}$, and $\Gamma \cap A$ is dense in A.

Lemma 5.4 *For every $\delta \in (0,1)$ there is a $\gamma \in \Gamma$ such that all eigenvalues of the automorphism $Ad_G(\gamma)\big|_A : A \to A$ have absolute value $< \delta$.*

Proof. See section 5.1 for terminology.
Step 1: A contains no semisimple isometries other than e. Otherwise we can consider the intersection C of the minimum sets for the displacement functions δ_a where a runs through all semisimple elements in A. C is a nonempty convex subset of X which splits metrically as $C \cong \mathbb{E}^k \times Y$. The flats $\mathbb{E}^k \times \{y\}$ are the minimal flats preserved by all semisimple elements in A. Since Γ normalises A it follows that C is Γ-invariant. The cocompactness of Γ implies that $C = X$ and $k = 0$ because X has no Euclidean factor. This means that the semisimple elements in A fix all points, a contradiction.
Step 2: All non-trivial isometries in A are purely parabolic. If $a \in A$, $a \neq e$, is not purely parabolic then there is a sequence of isometries g_n so that $g_n a g_n^{-1}$ converges to a semisimple isometry $\bar{a} \neq e$. We can uniformly approximate the g_n by elements in Γ, i.e. there exist $\gamma_n \in \Gamma$ and a bounded sequence $k_n \in G$ subconverging to $k \in G$ so that $\gamma_n = k_n g_n$. Then $\gamma_n a \gamma_n^{-1} = k_n g_n a g_n^{-1} k_n^{-1}$ subconverges to the non-trivial semisimple element $k \bar{a} k^{-1}$. This contradicts step 1.
Step 3: Pick a basis $\{a_1, \ldots, a_k\}$ for $A \simeq \mathbb{R}^k$. By lemma 5.2 there exist elements $g_n \in G$ so that $g_n a_i g_n^{-1} \to e$ for all a_i. We approximate the g_n as above by γ_n so that the sequence $\gamma_n g_n^{-1}$ is bounded. Then $\gamma_n a_i \gamma_n^{-1} \to e$ for all a_i. The lemma follows by setting $\gamma = \gamma_n$ for sufficiently large n. □

Proof of case 1 continued. By the lemma, there is a $\gamma \in \Gamma$, $\gamma \neq e$, and a norm $\|\cdot\|_A$ on A such that for all $a \in A$ we have

$$\|\gamma a \gamma^{-1}\|_A < \frac{1}{2}\|a\|_A.$$

Consider a neighborhood U of e in G. Let $r > 0$ be small enough so that $\{a \in A : \|a\|_A < r\} \subset U$ and pick $\alpha \in \Gamma \cap A$ with $\|\alpha\|_A < r/2$. Then the elements

$$\gamma_{\epsilon_0 \ldots \epsilon_{n-1}} = \alpha^{\epsilon_0} \cdot (\gamma \alpha \gamma^{-1})^{\epsilon_1} \cdot \cdots \cdot (\gamma^{n-1} \alpha \gamma^{1-n})^{\epsilon_{n-1}}$$

for $\epsilon_i \in \{0, 1\}$ are 2^n pairwise distinct elements contained in $\Gamma \cap U$ with word norm $|\gamma_{\epsilon_0 \ldots \epsilon_{n-1}}|_\Gamma < n^2(|\alpha|_\Gamma + |\gamma|_\Gamma)$. This implies superpolynomial growth of f.

Case 2: $\bar{\Gamma}^o$ is not nilpotent. Define an increasing sequence (the upper central series) of nilpotent Lie subgroups $Z_i \subset \bar{\Gamma}^o$ inductively as follows: Set $Z_0 = \{e\}$ and let Z_{i+1} be the inverse image in $\bar{\Gamma}^o$ of the center in $\bar{\Gamma}^o / Z_i$. The dimension of Z_i stabilizes and we choose k so that $\dim Z_k$ is maximal. Then the center of $\bar{\Gamma}/Z_k$ is discrete and, since $\bar{\Gamma}^o$ is not nilpotent, we have $\dim Z_k < \dim \bar{\Gamma}$. Proposition 5.3 now follows by applying the next lemma with $H = \bar{\Gamma}$ and $H_1 = Z_k$. □

Lemma 5.5 *Let H be a Lie group, let $H_1 \lhd H$ be a closed normal subgroup so that $\bar{H} := H/H_1$ is a positive dimensional Lie group with discrete center, and suppose $\Gamma \subset H$ is a dense, finitely generated subgroup. If U is any neighborhood of e in H, then the function $f(k) := \#\{g \in \Gamma : |g|_\Gamma \leq k,\ g \in U\}$ grows superpolynomially.*

Proof. The idea of the proof is to use the contracting property of commutators to produce a sequence $\{\alpha_k\}$ in $H \cap \Gamma$ which converges exponentially to the identity. The word norm $|\alpha_k|_\Gamma$ grows exponentially with k, but the number of elements of $\langle \alpha_1, \ldots, \alpha_k \rangle$ in U also grows exponentially with k; by comparing growth exponents we find that f grows superpolynomially.

Fix $M \in \mathbb{N}$, a positive real number $\epsilon < 1/3$ and some left-invariant Riemannian metric on H. Since the differential of the commutator map $(h, h') \mapsto [h, h']$ vanishes at (e, e) we can find a neighborhood V of e in H such that:

$$h, h' \in V \quad \Longrightarrow \quad [h, h'] \in V \quad \text{and} \quad d([h, h'], e) < \frac{1}{2M} d(h, e) \tag{3}$$

Since the differential of the k-th power $h \mapsto h^k$ at e is $k \cdot id_{T_e H}$ for all $k \in \mathbb{Z}$, we can furthermore acheive that, whenever $1 \leq k, k' \leq M$ and $h, h^k, h^{k'} \in V$, then

$$d(h^k, h^{k'}) \geq (|k - k'| - \epsilon) \cdot d(h, e) \tag{4}$$

By our assumption, there exist finitely many elements $\gamma_1, \dots, \gamma_m \in \Gamma \cap V$ such that the centralizers $Z_{\bar{H}}(\bar{\gamma}_j)$ of their images in $\bar{H}$ have discrete intersection. We construct an infinite sequence of elements $\alpha_i \in (\Gamma \cap V) \setminus H_1$ by picking $\alpha_0 \in V$ arbitrarily and setting $\alpha_{i+1} = [\alpha_i, h_{j(i)}] \notin H_1$ for suitably chosen $1 \leq j(i) \leq m$. Then

$$0 < d(\alpha_{i+1}, e) < \frac{1}{2M} d(\alpha_i, e) \tag{5}$$

by (3).

Sublemma 5.6 *Pick $n_0 \in \mathbb{N}$. The M^n elements*

$$\gamma_{\epsilon_1 \dots \epsilon_n} = \alpha_{n_0+1}^{\epsilon_1} \cdots \alpha_{n_0+n}^{\epsilon_n} \qquad \epsilon_i \in \{0, \dots, M-1\} \tag{6}$$

are distinct.

Proof. Assume that $\gamma_{\epsilon_1 \dots \epsilon_n} = \gamma_{\epsilon'_1 \dots \epsilon'_n}$, $\epsilon_l \neq \epsilon'_l$ and $\epsilon_i = \epsilon'_i$ for all $i < l$. Then

$$\alpha_{n_0+l}^{\epsilon_l - \epsilon'_l} = \alpha_{n_0+l+1}^{\epsilon'_{l+1} - \epsilon_{l+1}} \cdots \alpha_{n_0+n}^{\epsilon'_n - \epsilon_n}.$$

On the other hand (4,5) and the triangle inequality imply

$$d(\alpha_{n_0+l+1}^{\epsilon'_{l+1} - \epsilon_{l+1}} \cdots \alpha_{n_0+n}^{\epsilon'_n - \epsilon_n}, e) < M \cdot \sum_{j=1}^{\infty} \frac{1}{(2M)^j} \cdot d(\alpha_{n_0+l}, e) < \frac{1}{2} d(\alpha_{n_0+l}, e) < d(\alpha_{n_0+l}^{\epsilon_l - \epsilon'_l}, e),$$

a contradiction. $\square$

To complete the proof of the lemma, we observe that the elements (6) have word norm $|\gamma_{\epsilon_1 \dots \epsilon_n}|_\Gamma \leq const(n_0) \cdot 2^n$ and are contained in U if n_0 is sufficiently large. This shows that $f(k)$ grows polynomially of order at least $\frac{log(M)}{log(2)}$ for all M, hence the claim. $\square$

6 Proof of the main theorem

Proof of theorem 1.1: Let $\rho_0 : \Gamma \times \Gamma \to \Gamma$ be the isometric action of Γ on itself by left translation, and let $\phi : \Gamma \to Nil \times X$ be a quasi-isometry. Then there is a quasi-action ρ of Γ on $Nil \times X$ such that ϕ quasi-isometrically conjugates ρ_0 into ρ. According to section 3, ρ projects to a cobounded quasi-action $\bar{\rho}$ of Γ on X. After

quasi-isometrically conjugating ρ we may assume that $\bar{\rho}$ is an isometric action, cf. 4.5. For any point $y \in Nil \times X$ the map $\Gamma \to Nil \times X$ given by $\gamma \mapsto \rho(\gamma, y)$ is a quasi-isometry. Thus for all $x \in X$ and $R > 0$

$$\#\{\gamma \in \Gamma \mid |\gamma|_\Gamma < k,\ \bar{\rho}(\gamma) \cdot x \in B_R(x)\}$$

grows at most as fast as the volume of balls in Nil, i.e. $< k^d$ for some d. Proposition 5.3 implies that $L := \bar{\rho}(\Gamma)$ is a discrete subgroup in $Isom(X)$ and hence a uniform lattice. The kernel H of the action $\bar{\rho}$ is then quasi-isometric to the fiber Nil. $\square$

References

[CJ94] A. Casson and D. Jungreis, *Convergence groups and Seifert fibered 3-manifolds,* Inv. Math. **118** no. 3 (1994), 441–456.

[Chow96] R. Chow, *Groups quasi-isometric to complex hyperbolic space*, Trans. AMS, **348** (1996), no. 5, pp. 1757-1769.

[Esk96] A. Eskin, *Quasi-isometric rigidity of nonuniform lattices in higher rank symmetric spaces,* Preprint 1996.

[Ga92] D. Gabai, *Convergence groups are Fuchsian groups,* Ann. Math. **136** no. 3 (1992), 447–510.

[Ger92] S. Gersten, *Bounded cocycles and combings of groups*, Int. J. Alg. Comp., **2**, (1992), no. 3, 307-326.

[GDH90] E. Ghys, P. de la Harpe, *Sur les groupes hyperboliques d'apres Mikhael Gromov*, Progress in Mathematics, **83**, Birkhauser.

[Gro93] M. Gromov, *Asymptotic invariants for infinite groups*, in: Geometric group theory, London Math. Soc. lecture note series 182, 1993.

[Gro81] M. Gromov, *Hyperbolic manifolds, groups, and actions*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Confernce, 183-213, Ann. of Math. Stud. 97.

[Gro81'] M. Gromov, *Groups of polynomial growth and expanding maps*, Publ. IHES, **53** (1981), pp. 53-73.

[Gro87] M. Gromov, *Hyperbolic groups*, 75-263, In: Essays in group theory, MSRI Publ. 8, Springer, 1987.

[Hin90] A. Hinkkanen, *The structure of certain quasi-symmetric groups*, Mem. Amer. Math. Soc., **83**, (1990), no. 422, 1-83.

[KaLe96] M. Kapovich and B. Leeb, *Quasi-isometries preserve the geometric decomposition of Haken manifolds,* to appear in Inventiones Math.

[KlLe96] B. Kleiner and B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings,* Preprint, 1996.

[Pan83] P. Pansu, *Croissance des boules et des géodésiques fermées dans les nil-variétés*, Erg. Thy. Dyn. Sys., **3**, (1983), no. 3, 415-445.

[Pan89] P. Pansu, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symetriques de rang un,* Ann. of Math., **129** (1989), 1–60.

[Rie93] E. Rieffel, *Groups coarse quasi-isometric to* $\mathbb{H}^2 \times \mathbb{R}$, PhD Thesis, UCLA, 1993.

[Sch95] R. Schwartz, *The quasi-isometry classification of rank one lattices,* Publ. of IHES, vol. **82** (1995) 133– 168.

[Tuk88] P. Tukia, *Homeomorphic conjugates of Fuchsian groups,* J. Reine Angew. Math. **391** (1988), 1–54.

[TuVa82] P. Tukia, J. Väisälä, *Quasiconformal extension from dimension* n *to* $n+1$, Ann. Math., **115**, (1982), 331-348.

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